New Spinor Field Realizations of the Non-Critical W_3 String*

Li-Jie Zhang

[1] *Department of Mathematics and Physics,*

Dalian Jiaotong University, Dalian 116028

Yu-Xiao Liu† and Ji-Rong Ren

[2] *Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000*

Abstract

We investigate the new spinor field realizations of the W_3 algebra, making use of the fact that the W_3 algebra can be linearized by the addition of a spin-1 current. We then use these new realizations to build the nilpotent Becchi-Rouet-Stora–Tyutin (BRST) charges of the spinor non-critical W_3 string.

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†Corresponding author; Electronic address: liuyx01@st.lzu.edu.cn

As is well known, W algebra has remarkable applications in W gravity and W string theories since its discovery in the 1980s.[1,2] Furthermore, it appears in the quantum Hall effect, black holes, in lattice models of statistical mechanics at criticality, and in other physical models.[3]

The Becchi-Rouet-Stora–Tyutin (BRST) formalism[4] has proven to be rather fruitful in the study of the critical and non-critical W string theories. The BRST charge of W_3 (i.e. $W_{2,3}$) string was first constructed in Ref. [5], and its detailed studies can be found in Refs. [5-7]. A natural generalization of the W_3 string, i.e. the $W_{2,s}$ strings, is a higher-spin string with local spin-2 and spin-s symmetries on the world-sheet. Many works have been carried out on the scalar field realizations of $W_{2,s}$ strings.[7−9] The BRST charges for such theories have been constructed for $s = 4, 5, 6, 7$.[8] Later we discovered the reason that the scalar BRST charge is difficult to be generalized to a general W_N string.[10] At the same time, we found the methods to construct the spinor field realization of $W_{2,s}$ strings and W_N strings.[10] Subsequently, we studied the exact spinor field realizations of $W_{2,s}(s = 3, 4, 5, 6)$ strings and $W_N(N = 4, 5, 6)$ strings.[10,11] Recently, we have constructed the nilpotent BRST charges of spinor non-critical $W_{2,s}(s = 3, 4)$ strings by taking into account the property of spinor field.[12] These results will be important for constructing super W strings, and they will provide the essential ingredients.

However, all of these theories about the $W_{2,s}$ strings mentioned above are based on the non-linear $W_{2,s}$ algebras. Because of the intrinsic nonlinearity of the $W_{2,s}$ algebras, their study is a more difficult task compared to linear algebras. Fortunately, it has been shown that certain W algebras can be linearized by the inclusion of a spin-1 current. This provides a way of obtaining new realizations of the $W_{2,s}$ algebras. Such new realizations were constructed for the purpose of building the corresponding scalar $W_{2,s}$ strings.[13]

Since there has been no work focused on the research of spinor field realizations of the non-critical W_3 string based on the linear $W_{1,2,3}$ algebra, in this Letter we construct new nilpotent BRST charges of spinor non-critical W_3 string for the first time by using the linear bases of the $W_{1,2,3}$ algebra. To construct a non-critical BRST charge, one must first solve the forms of matter currents T and W determined by the OPEs of TT, TW and WW. Here T and W here are constructed on the linear bases T_0, J_0, W_0 of the $W_{1,2,3}$ algebra, and these linear bases are constructed with the spinor fields. Then direct substitution of these results into BRST charge leads to the grading spinor field realizations. All these results will be

important for embedding the Virasoro string into the W_3 string.

We begin by reviewing the linearization of the W_3 algebra by the inclusion of a spin-1 current.[14] We take the linearized $W_{1,2,3}$ algebra in the form

$$\begin{aligned}
T_0(z)T_0(\omega) &\sim \frac{C/2}{(z-w)^4}+\frac{2T_0(\omega)}{(z-\omega)^2}+\frac{\partial T_0(\omega)}{z-\omega},\\
T_0(z)W_0(\omega) &\sim \frac{3W_0(\omega)}{(z-\omega)^2}+\frac{\partial W_0(\omega)}{z-\omega},\\
T_0(z)J_0(\omega) &\sim \frac{C_1}{(z-w)^3}+\frac{J_0(\omega)}{(z-\omega)^2}+\frac{\partial J_0(\omega)}{z-\omega},\\
J_0(z)J_0(\omega) &\sim \frac{-1}{(z-\omega)^2},\\
J_0(z)W_0(\omega) &\sim \frac{hW_0(\omega)}{z-\omega},\quad W_0(z)W_0(\omega)\sim 0.
\end{aligned}\tag{1}$$

The coefficients C, C_1 and h are given by

$$C=50+24t^2+\frac{24}{t^2},\ C_1=-\sqrt{6}(t+\frac{1}{t}),\ h=\sqrt{\frac{3}{2}}\,t.\tag{2}$$

To obtain the new realizations for the linearized $W_{1,2,3}$ algebra, we use the multi-spinor fields ψ^μ, which have spin 1/2 and satisfy the OPE

$$\psi^\mu(z)\psi^\nu(\omega)\sim-\frac{1}{z-\omega}\,\delta^{\mu\nu},$$

for the first time to construct the linear bases of them. The general forms of these linear bases can be taken as follows:

$$T_0=-\frac{1}{2}\partial\psi^\mu\psi^\mu,\ J_0=\alpha_{\mu\nu}\psi^\mu\psi^\nu\ (\mu<\nu),\ W_0=0,$$

where $\alpha_{\mu\nu}$ are pending coefficients. By making use of the OPE $J_0(z)J_0(\omega)$ in Eq. (1), we can obtain the equation that the coefficients $\alpha_{\mu\nu}$ satisfy, i.e. $\sum_{\mu<\nu}\alpha_{\mu\nu}^2=1$. From the OPE relation of T_0 and J_0, it is easy to obtain $C_1=0$. Substituting the value of C_1 into Eq. (2), we can obtain the value of t. Then the total central charge C for T_0 can be obtained from Eq. (2). Thus we can determine the explicit form of T_0 under the restricted condition of its central charge. Finally, using the OPE $T_0(z)J_0(\omega)$ in Eq. (1) again, we reach the exact form of J_0. The complete results are $t=\pm i,\ C=2,\ C_1=0$ and

$$T_0=-\frac{1}{2}\sum_{\mu=1}^{4}\partial\psi^\mu\psi^\mu,\ J_0=\sum_{\mu<\nu=1}^{4}\alpha_{\mu\nu}\psi^\mu\psi^\nu,\ W_0=0,\tag{3}$$

where the coefficients $\alpha_{\mu\nu}$ satisfy

$$\sum_{\mu<\nu=1}^{4}(\alpha_{\mu\nu})^2 = 1. \tag{4}$$

As we know, for the energy-momentum tensor of a scalar ϕ, it has central charge $C = 1$ in two-dimensional conformal field theory, while $C = 1/2$ for a single fermion field. This means that one real scalar field is equivalent to two real or one complex fermion fields. In particular, they correspond to the forms $\psi =: e^{i\phi} :, \psi = \psi_1 + i\psi_2$. Similarly, the power exponent can also be used as a means of bosonisation in our constructions. Then, our 4-fermion construction is equivalent to a two-scalar system.

Now let us consider the spinor realizations of the non-linear W_3 algebra with linear bases of the $W_{1,2,3}$ algebra. We begin by reviewing the structures of the W_3 algebra in conformal language. The OPE $W(z)W(\omega)$ for W_3 algebra is given by[1]

$$\begin{aligned} W(z)W(\omega) \sim\ & \frac{C/3}{(z-w)^6} + \frac{2T}{(z-\omega)^4} + \frac{\partial T}{(z-\omega)^3} \\ & + \frac{1}{(z-\omega)^2}\left(2\Theta\Lambda + \frac{3}{10}\partial^2 T\right) \\ & + \frac{1}{(z-\omega)}\left(\Theta\partial\Lambda + \frac{1}{15}\partial^3 T\right), \end{aligned} \tag{5}$$

where

$$\Theta = \frac{16}{22+5C}, \quad \Lambda = T^2 - \frac{3}{10}\partial^2 T.$$

The bases of the W_3 algebra can be constructed by the linear bases of the $W_{1,2,3}$ algebra:

$$T = T_0,\ W = W_0 + W_R(J_0, T_0),$$

where the currents $T_0,\ J_0$ and W_0 generate the $W_{1,2,3}$ algebra and have been constructed with multi-spinor fields. First we can write the most general possible structure of W. Then the relations of the above OPEs of T and W determine the coefficients of the terms in W. Finally, substituting these coefficients together with $T_0,\ J_0$ and W_0 into the expressions of T and W, we can obtain the spinor realizations of the W_3 algebra. The explicit results turn out to be very simple:

$$T = T_0,\ W = W_0 \pm \frac{1}{6}(-3i\partial^2 J_0 + 4iJ_0^3 + 6iT_0 J_0). \tag{6}$$

Substituting the expressions of T_0, J_0 and W_0 in Eq. (3) into Eq. (6), we obtain the explicit constructions of T and W for the W_3 algebra as follows:

$$
\begin{aligned}
T &= -\frac{1}{2}\sum_{\mu=1}^{4}\partial\psi^{\mu}\psi^{\mu}, && (7)\\
W &= \pm\frac{1}{2i}\left(\sum_{\mu<\nu,\lambda,\rho=1}^{4}|\epsilon_{\mu\nu\lambda\rho}|\alpha_{\mu\nu}\psi^{\mu}\psi^{\nu}\partial\psi^{\lambda}\psi^{\lambda}\right.\\
&\quad\left.+\sum_{\mu<\nu=1}^{4}\alpha_{\mu\nu}(\partial^2\psi^{\mu}\psi^{\nu}+\psi^{\mu}\partial^2\psi^{\nu}+2\partial\psi^{\mu}\partial\psi^{\nu})\right), && (8)
\end{aligned}
$$

where $\alpha_{\mu\nu}$ satisfies Eq. (4).

The non-critical W_3 string is the theory of W_3 gravity coupled to a matter system on which the W_3 algebra is realized. Subsequently, we give realizations of the spinor field .

The BRST charges of the non-critical W_3 string take the form:

$$
\begin{aligned}
Q_B &= Q_0 + Q_1, && (9)\\
Q_0 &= \oint dz\, c(T_{\psi}+T_M+T_{bc}+yT_{\beta\gamma}+T^{eff}), && (10)\\
Q_1 &= \oint dz\, \gamma F(\psi,\beta,\gamma,T_M,W_M), && (11)
\end{aligned}
$$

where y is a pending constant. The matter currents T_M and W_M, which have spin 2 and 3 respectively, generate the W_3 algebra and have been constructed for the cases $s=3$ (T and W in Eqs. (7) and (8) are replaced by T_M and W_M, respectively). The energy-momentum tensors in Eq. (10) are given by[12,15]

$$
\begin{aligned}
T_{\psi} &= -\frac{1}{2}\partial\psi\psi,\ T_M = T,\ T_{bc} = 2b\partial c + \partial bc,\\
T_{\beta\gamma} &= 3\beta\partial\gamma + 2\partial\beta\gamma,\ T^{eff} = -\frac{1}{2}\eta_{\mu\nu}\partial Y^{\mu}Y^{\nu}.
\end{aligned}
$$

The operator $F(\psi,\beta,\gamma,T_M,W_M)$ has the spin of 3 and the ghost number of zero. The BRST charge generalizes the one for scalar non-critical $W_{2,s}$ strings, and it is also graded with $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. Since the first condition is satisfied for any s automatically, the remaining two conditions determine y and the coefficients of the terms in $F(\psi,\beta,\gamma,T_M,W_M)$.

Now, using the grading BRST method and the procedure mentioned above, we discuss the exact solutions of spinor field realizations of the non-critical W_3 string.

In this case, Q_B takes the form of Eq. (9). The mater currents T_M and W_M are given by Eqs. (7) and (8), respectively. The most extensive combinations of F in Eq. (11) with

correct spin and ghost number can be constructed as follows:

$$
\begin{aligned}
F \;=\; & f_1\beta^3\gamma^3 + f_2\partial\beta\beta\gamma^2 + f_3\partial\beta\partial\gamma + f_4\beta\gamma\partial\psi\psi \\
& + f_5\beta\partial^2\gamma + f_6\partial^2\psi\psi + f_7\beta\gamma T_M \\
& + f_8\partial\psi\psi T_M + f_9\partial T_M + f_{10}W_M.
\end{aligned} \tag{12}
$$

Substituting Eq. (12) into Eq. (11) and imposing the nilpotency conditions $Q_1^2 = \{Q_0, Q_1\} = 0$, we can determine y and $f_i (i = 1, 2, \cdots, 10)$. They correspond to three sets of general solutions, i.e.

(i) $y = 0$, $f_i = 0$ $(i = 4, 6, 7, 8, 9, 10)$, and f_j $(j = 1, 2, 3, 5)$ are arbitrary constants but do not vanish at the same time.

(ii) $y = 1$ and

$$
\begin{aligned}
& f_1 = \frac{1}{150}(-7f_3 + 3f_5),\;\; f_2 = \frac{1}{15}(7f_3 - 3f_5), \\
& f_4 = \frac{1}{5}(22f_3 - 78f_5 + 10f_7), \\
& f_6 = -11f_3 + 39f_5 - 5f_7,\;\; f_8 = 0,\;\; f_9 = -\frac{5}{2}f_7,
\end{aligned}
$$

where f_j $(j = 3, 5, 7, 10)$ are arbitrary constants but do not vanish at the same time.

(iii) y is an arbitrary constant and

$$
\begin{aligned}
f_i \;&=\; 0 \quad (i = 4, 6, 7, 8, 9, 10), \\
f_1 \;&=\; -\frac{8}{55}f_5,\;\; f_2 = \frac{16}{11}f_5,\;\; f_3 = \frac{39}{11}f_5,
\end{aligned}
$$

where f_5 is an arbitrary non-zero constant.

Substituting the construction (12) of F into Eq. (11) and using T_M and W_M given by Eqs. (7) and (8), we obtain the explicit spinor field realizations of the non-critical W_3 string.

In summary, we have constructed the new spinor field realizations of the non-critical W_3 string, making use of the fact that the W_3 algebra can be linearized by the addition of a spin-1 current. First, we use the multi-spinor fields ψ^μ to construct the linear bases of the linearized $W_{1,2,3}$ algebra. Subsequently, the non-linear bases of W_3 algebra is constructed with these linear bases T_0, J_0 and W_0. Finally, we use these new realizations to build the graded BRST charges of the spinor non-critical W_3 string. The constructions are based on demanding the nilpotency of the BRST charges. The solutions are very standard, i.e., there are three solutions for the W_3 string. It is worth noting that our four-fermion result

has central charge $C = 2$, while for the critical W_3 string the central charge should be $C = 100$. Since the (T_0, J_0) system is linear, we can add more matter to obtain $C = 100$. For simplicity, considering the independence of central charge under non-critical case, we only give the discussion based on $C = 2$ in this study. We expect that there should exist such realizations for the case of higher spin s. Having obtained the exact BRST charges of W_3 string, we can investigate the implications for the corresponding string theories.

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